ON A CONJECTURE REGARDING ENUMERATION OF N-TIMES PERSYMMETRIC MATRICES OVER $\mathbb{F}_2$ BY RANK

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Résumé. Dans cet article nous annonçons une conjecture concernant l'énumération de n- fois matrices persymétriques sur $\mathbb{F}_2$ par le rang. Pour justifier notre assertion nous faisons remarquer que les formules obtenues sont valables pour n égal à un, deux et trois.

Abstract. In this paper we announce a conjecture concerning enumeration of n-times persymmetric matrices over $\mathbb{F}_2$ by rank. To justify our statement we remark that the formulas obtained are valid for n equal to one, two and three.

Contents

1. Some notations concerning the field of Laurent Series $\mathbb{F}_2((T^{-1}))$

We denote by $\mathbb{F}_2\big(\big(T^{-1}\big)\big) = \mathbb{K}$ the completion of the field $\mathbb{F}_2(T)$, the field of rational fonctions over the finite field $\mathbb{F}_2$, for the infinity valuation $\mathfrak{v} = \mathfrak{v}_\infty$ defined by $\mathfrak{v}\big(\frac{A}{B}\big) = degB - degA$ for each pair (A,B) of non-zero polynomials. Then every element non-zero t in $\mathbb{F}_2\big(\big(\frac{1}{T}\big)\big)$ can be expanded in a unique way in a convergent Laurent series $t = \sum_{j=-\infty}^{-\mathfrak{v}(t)} t_j T^j \; where \; t_j \in \mathbb{F}_2$.
We associate to the infinity valuation $\mathfrak{v} = \mathfrak{v}_\infty$ the absolute value $|\cdot|_\infty$ defined by

$$|t|_\infty = |t| = 2^{-\mathfrak{v}(t)}.$$

We denote E the Character of the additive locally compact group $\mathbb{F}_2\big(\big(\frac{1}{T}\big)\big)$ defined by

$$E\Big(\sum_{j=-\infty}^{-\mathfrak{v}(t)} t_j T^j\Big) = \begin{cases} 1 & \text{if } t_{-1} = 0, \\ -1 & \text{if } t_{-1} = 1. \end{cases}$$

We denote $\mathbb{P}$ the valuation ideal in $\mathbb{K}$, also denoted the unit interval of $\mathbb{K}$, i.e. the open ball of radius 1 about 0 or, alternatively, the set of all Laurent series

$$\sum_{i\geq 1} \alpha_i T^{-i} \quad (\alpha_i \in \mathbb{F}_2)$$

and, for every rational integer j, we denote by $\mathbb{P}_j$ the ideal $\{t \in \mathbb{K}|\ \mathfrak{v}(t) > j\}$.
The sets $\mathbb{P}_j$ are compact subgroups of the additive locally compact group $\mathbb{K}$.
All $t \in \mathbb{F}_2\Big(\Big(\frac{1}{T}\Big)\Big)$ may be written in a unique way as $t = [t] + \{t\}$, $[t] \in \mathbb{F}_2[T]$, $\{t\} \in \mathbb{P}(= \mathbb{P}_0)$.
We denote by dt the Haar measure on $\mathbb{K}$ chosen so that

$$\int_{\mathbb{P}} dt = 1.$$

$$Let \quad (t_1, t_2, \ldots, t_n) = \Big(\sum_{j=-\infty}^{-\nu(t_1)} \alpha_j^{(1)} T^j, \sum_{j=-\infty}^{-\nu(t_2)} \alpha_j^{(2)} T^j, \ldots, \sum_{j=-\infty}^{-\nu(t_n)} \alpha_j^{(n)} T^j\Big) \in \mathbb{K}^n.$$

We denote ψ the Character on $(\mathbb{K}^n, +)$ defined by

$$\begin{aligned}\psi\Big(\sum_{j=-\infty}^{-\nu(t_1)} \alpha_j^{(1)} T^j, \sum_{j=-\infty}^{-\nu(t_2)} \alpha_j^{(2)} T^j, \ldots, \sum_{j=-\infty}^{-\nu(t_n)} \alpha_j^{(n)} T^j\Big) &= E\Big(\sum_{j=-\infty}^{-\nu(t_1)} \alpha_j^{(1)} T^j\Big) \cdot E\Big(\sum_{j=-\infty}^{-\nu(t_2)} \alpha_j^{(2)} T^j\Big) \cdots E\Big(\sum_{j=-\infty}^{-\nu(t_n)} \alpha_j^{(n)} T^j\Big) \\ &= \begin{cases} 1 & \text{if } \alpha_{-1}^{(1)} + \alpha_{-1}^{(2)} + \ldots + \alpha_{-1}^{(n)} = 0 \\ -1 & \text{if } \alpha_{-1}^{(1)} + \alpha_{-1}^{(2)} + \ldots + \alpha_{-1}^{(n)} = 1 \end{cases}\end{aligned}$$

1.1. Computation of the number $\Gamma_{i}^{\left[\begin{smallmatrix}1\\ \vdots\\ 1\end{smallmatrix}\right]\times 2}$ of n-times persymmetric $n\times 2$ rank i matrices.

$$Set \quad (t_1,t_2,\ldots,t_n) = \big(\sum_{i\geq 1}\alpha_i^{(1)}T^{-i},\sum_{i\geq 1}\alpha_i^{(2)}T^{-i},\sum_{i\geq 1}\alpha_i^{(3)}T^{-i},\ldots,\sum_{i\geq 1}\alpha_i^{(n)}T^{-i}\big)\in\mathbb{P}^n.$$

We denote by $D^{\left[\begin{smallmatrix}1\\ \vdots\\ 1\end{smallmatrix}\right]\times 2}(t_1,t_2,\ldots,t_n)$ the following $n\times 2$ n-times persymmetric matrix over the finite field $\mathbb{F}_2$

$$\left(\begin{array}{cc} \alpha_1^{(1)} & \alpha_2^{(1)} \\ \hline \alpha_1^{(2)} & \alpha_2^{(2)} \\ \hline \alpha_1^{(3)} & \alpha_2^{(3)} \\ \hline \vdots & \vdots \\ \hline \alpha_1^{(n)} & \alpha_2^{(n)} \end{array}\right) \overset{\text{rank}}{\sim} \left(\begin{array}{cccc} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \beta_1 & \beta_2 & \ldots & \beta_n \end{array}\right)$$

Let $f(t_1,t_2,\ldots,t_n)$ be the exponential sum in $\mathbb{P}^n$ defined by
$(t_1,t_2,\ldots,t_n)\in\mathbb{P}^n \longrightarrow$

$$\sum_{degY\leq 1}\sum_{degU_1\leq 0}E(t_1YU_1)\sum_{degU_2\leq 0}E(t_2YU_2)\ldots\sum_{degU_n\leq 0}E(t_nYU_n).$$

Then

$$f(t_1,t_2,\ldots,t_n) = 2^{n+2-rank\left[D^{\left[\begin{smallmatrix}1\\ \vdots\\ 1\end{smallmatrix}\right]\times 2}(t_1,t_2,\ldots,t_n)\right]}$$

Hence the number denoted by R_q of solutions

$(Y_1,U_1^{(1)},U_2^{(1)},\ldots,U_n^{(1)},Y_2,U_1^{(2)},U_2^{(2)},\ldots,U_n^{(2)},\ldots Y_q,U_1^{(q)},U_2^{(q)},\ldots,U_n^{(q)})\in(\mathbb{F}_2[T])^{(n+1)q}$

of the polynomial equations

$$\begin{cases} Y_1U_1^{(1)}+Y_2U_1^{(2)}+\ldots+Y_qU_1^{(q)}=0\\ Y_1U_2^{(1)}+Y_2U_2^{(2)}+\ldots+Y_qU_2^{(q)}=0\\ \qquad\qquad\vdots\\ Y_1U_n^{(1)}+Y_2U_n^{(2)}+\ldots+Y_qU_n^{(q)}=0 \end{cases}$$

satisfying the degree conditions

$$degY_i\leq 1,\quad degU_j^{(i)}\leq 0,\quad for\quad 1\leq j\leq n\quad 1\leq i\leq q$$

is equal to the following integral over the unit interval in $\mathbb{K}^n$

$$\int_{\mathbb{P}^n} f^q(t_1,t_2,\ldots,t_n)dt_1dt_2\ldots dt_n.$$

Observing that $f(t_1,t_2,\ldots,t_n)$ is constant on cosets of $\mathbb{P}_2^n$ in $\mathbb{P}^n$ the above integral is equal to

$$2^{q(n+2)-2n}\sum_{i=0}^{2}\Gamma_i^{\left[\begin{smallmatrix}1\\ \vdots\\ 1\end{smallmatrix}\right]\times 2} 2^{-iq} = R_q \tag{1.1}$$

From (1.1) we obtain for q = 1

$$2^{2-n}\sum_{i=0}^{2}\Gamma_i^{\left[\begin{smallmatrix}1\\ \vdots\\ 1\end{smallmatrix}\right]\times 2} 2^{-i} = 2^n+2^2-1 \tag{1.2}$$

We have obviously

$$\sum_{i=0}^{2}\Gamma_i^{\left[\begin{smallmatrix}1\\ \vdots\\ 1\end{smallmatrix}\right]\times 2} = 2^{2n} \tag{1.3}$$

Combining (1.2), (1.3) we get

$$\Gamma_i^{\left[\begin{smallmatrix}1\\ \vdots\\ 1\end{smallmatrix}\right]\times 2} = \begin{cases} 1 & \text{if } i=0,\\ (2^n-1)\cdot 3 & \text{if } i=1,\\ 2^{2n}-3\cdot 2^n+2 & \text{if } i=2 \end{cases} \tag{1.4}$$

From (1.1), (1.4) we obtain :

$$R_q = 2^{(q-2)n}\cdot[2^{2q}+2^{2n}+3\cdot(2^{n+q}-2^q-2^n)+2]$$

1.2. **Computation of the number $\Gamma_i^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times 3}$ of n-times persymmetric $2n\times 3$ rank i matrices.**

Set $(t_1,t_2,\ldots,t_n) = \big(\sum_{i\geq 1}\alpha_i^{(1)}T^{-i},\sum_{i\geq 1}\alpha_i^{(2)}T^{-i},\sum_{i\geq 1}\alpha_i^{(3)}T^{-i},\ldots,\sum_{i\geq 1}\alpha_i^{(n)}T^{-i}\big)\in\mathbb{P}^n.$

We denote by $D^{\left[\begin{smallmatrix}2\\ \vdots \\ 2\end{smallmatrix}\right]\times 3}(t_1,t_2,\ldots,t_n)$ the following $2n\times 3$ n-times persymmetric matrix over the finite field $\mathbb{F}_2$

$$\left(\begin{array}{ccc}
\alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} \\
\alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} \\
\hline
\alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} \\
\alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} \\
\hline
\alpha_1^{(3)} & \alpha_2^{(3)} & \alpha_3^{(3)} \\
\alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} \\
\hline
\vdots & \vdots & \vdots \\
\hline
\alpha_1^{(n)} & \alpha_2^{(n)} & \alpha_3^{(n)} \\
\alpha_2^{(n)} & \alpha_3^{(n)} & \alpha_4^{(n)}
\end{array}\right)$$

Let $f(t_1,t_2,\ldots,t_n)$ be the exponential sum in $\mathbb{P}^n$ defined by
$(t_1,t_2,\ldots,t_n)\in\mathbb{P}^n \longrightarrow$
$\sum_{degY\leq 2}\sum_{degU_1\leq 1}E(t_1YU_1)\sum_{degU_2\leq 1}E(t_2YU_2)\ldots\sum_{degU_n\leq 1}E(t_nYU_n).$

Then

$$f(t_1,t_2,\ldots,t_n)=2^{2n+3-rank\left[D^{\left[\begin{smallmatrix}2\\ \vdots \\ 2\end{smallmatrix}\right]\times 3}(t_1,t_2,\ldots,t_n)\right]}$$

Hence the number denoted by R_q of solutions

$(Y_1,U_1^{(1)},U_2^{(1)},\ldots,U_n^{(1)},Y_2,U_1^{(2)},U_2^{(2)},\ldots,U_n^{(2)},\ldots Y_q,U_1^{(q)},U_2^{(q)},\ldots,U_n^{(q)})\in(\mathbb{F}_2[T])^{(n+1)q}$

of the polynomial equations

$$\begin{cases} Y_1U_1^{(1)}+Y_2U_1^{(2)}+\ldots+Y_qU_1^{(q)}=0 \\ Y_1U_2^{(1)}+Y_2U_2^{(2)}+\ldots+Y_qU_2^{(q)}=0 \\ \vdots \\ Y_1U_n^{(1)}+Y_2U_n^{(2)}+\ldots+Y_qU_n^{(q)}=0 \end{cases}$$

satisfying the degree conditions

$$degY_i\leq 2,\quad degU_j^{(i)}\leq 1,\quad for\quad 1\leq j\leq n\quad 1\leq i\leq q$$

is equal to the following integral over the unit interval in $\mathbb{K}^n$

$$\int_{\mathbb{P}^n} f^q(t_1, t_2, \ldots, t_n) dt_1 dt_2 \ldots dt_n.$$

Observing that $f(t_1, t_2, \ldots, t_n)$ is constant on cosets of $\mathbb{P}_4^n$ in $\mathbb{P}^n$ the above integral is equal to

$$2^{q(2n+3)-4n} \sum_{i=0}^{3} \Gamma_i^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right] \times 3} 2^{-iq} = R_q \tag{1.5}$$

From (1.5) we obtain for q = 1

$$2^{3-2n} \sum_{i=0}^{3} \Gamma_i^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right] \times 3} 2^{-i} = 2^{2n} + 2^3 - 1 \tag{1.6}$$

We have obviously

$$\sum_{i=0}^{3} \Gamma_i^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right] \times 3} = 2^{4n} \tag{1.7}$$

From the fact that the number of rank one persymmetric matrices over $\mathbb{F}_2$ is equal to three we obtain using combinatorial methods that:

$$\Gamma_1^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right] \times 3} = (2^n - 1) \cdot 3 \tag{1.8}$$

Combining (1.6), (1.7) and (1.8) we get

$$\Gamma_i^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right] \times 3} = \begin{cases} 1 & \text{if } i = 0, \\ (2^n - 1) \cdot 3 & \text{if } i = 1, \\ 7 \cdot 2^{2n} - 9 \cdot 2^n + 2 & \text{if } i = 2, \\ 2^{4n} - 7 \cdot 2^{2n} + 6 \cdot 2^n & \text{if } i = 3 \end{cases} \tag{1.9}$$

Generalization:

Let $s_j \geqslant 2$ for $1 \leqslant j \leqslant n$, denote by $D^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_n \end{smallmatrix}\right] \times 3}(t_1, t_2, \ldots, t_n)$
the following $(\sum_{j=1}^{n} s_j) \times 3$ n-times persymmetric matrix over the finite field $\mathbb{F}_2$

$$\left(\begin{array}{ccc}
\alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} \\
\alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} \\
\vdots & \vdots & \vdots \\
\alpha_{s_1}^{(1)} & \alpha_{s_1+1}^{(1)} & \alpha_{s_1+2}^{(1)} \\
\hline
\alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} \\
\alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} \\
\vdots & \vdots & \vdots \\
\alpha_{s_2}^{(2)} & \alpha_{s_2+1}^{(2)} & \alpha_{s_2+2}^{(2)} \\
\hline
\alpha_1^{(3)} & \alpha_2^{(3)} & \alpha_3^{(3)} \\
\alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} \\
\vdots & \vdots & \vdots \\
\alpha_{s_3}^{(3)} & \alpha_{s_3+1}^{(3)} & \alpha_{s_3+2}^{(3)} \\
\hline
\vdots & \vdots & \vdots \\
\hline
\alpha_1^{(n)} & \alpha_2^{(n)} & \alpha_3^{(n)} \\
\alpha_2^{(n)} & \alpha_3^{(n)} & \alpha_4^{(n)} \\
\vdots & \vdots & \vdots \\
\alpha_{s_n}^{(n)} & \alpha_{s_n+1}^{(n)} & \alpha_{s_n+2}^{(n)}
\end{array}\right)$$

Let $f(t_1, t_2, \ldots, t_n)$ be the exponential sum in $\mathbb{P}^n$ defined by
$(t_1, t_2, \ldots, t_n) \in \mathbb{P}^n \longrightarrow$
$\sum_{degY \leq 2} \sum_{degU_1 \leq s_1-1} E(t_1YU_1) \sum_{degU_2 \leq s_2-1} E(t_2YU_2) \ldots \sum_{degU_n \leq s_n-1} E(t_nYU_n).$

Then

$$f(t_1, t_2, \ldots, t_n) = 2^{\sum_{i=1}^{n} s_i+3-rank\left[D^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_n \end{smallmatrix}\right]\times 3}_{(t_1,t_2,\ldots,t_n)}\right]}$$

Hence the number denoted by R_q of solutions

$$(Y_1, U_1^{(1)}, U_2^{(1)}, \ldots, U_n^{(1)}, Y_2, U_1^{(2)}, U_2^{(2)}, \ldots, U_n^{(2)}, \ldots Y_q, U_1^{(q)}, U_2^{(q)}, \ldots, U_n^{(q)})$$

of the polynomial equations

$$\begin{cases} Y_1U_1^{(1)} + Y_2U_1^{(2)} + \ldots + Y_qU_1^{(q)} = 0 \\ Y_1U_2^{(1)} + Y_2U_2^{(2)} + \ldots + Y_qU_2^{(q)} = 0 \\ \vdots \\ Y_1U_n^{(1)} + Y_2U_n^{(2)} + \ldots + Y_qU_n^{(q)} = 0 \end{cases}$$

satisfying the degree conditions

$$degY_i \leq 2, \quad degU_j^{(i)} \leq s_j - 1, \quad for \quad 1 \leq j \leq n \quad 1 \leq i \leq q$$

is equal to the following integral over the unit interval in $\mathbb{K}^n$

$$\int_{\mathbb{P}^n} f^q(t_1, t_2, \ldots, t_n) dt_1 dt_2 \ldots dt_n.$$

Observing that $f(t_1, t_2, \ldots, t_n)$ is constant on cosets of $\prod_{j=1}^{n} \mathbb{P}_{s_j+2}$ in $\mathbb{P}^n$ the above integral is equal to

$$2^{q(\sum_{j=1}^{n} s_j+3)-\sum_{j=1}^{n} s_j-2\cdot n} \sum_{i=0}^{3} \Gamma_i^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_n \end{smallmatrix}\right]\times 3} 2^{-iq} = R_q \tag{1.10}$$

From (1.10) we obtain for q = 1

$$2^{3-2n} \sum_{i=0}^{3} \Gamma_i^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_n \end{smallmatrix}\right]\times 3} 2^{-i} = 2^{\sum_{j=1}^{n} s_j} + 2^3 - 1 \tag{1.11}$$

We have obviously

$$\sum_{i=0}^{3} \Gamma_i^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_1 \end{smallmatrix}\right]\times 3} = 2^{\sum_{j=1}^{n} s_j+2n} \tag{1.12}$$

From the fact that the number of rank one persymmetric matrices over $\mathbb{F}_2$ is equal to three we obtain as above using combinatorial methods :

$$\Gamma_1^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_n \end{smallmatrix}\right]\times 3} = (2^n - 1) \cdot 3 \tag{1.13}$$

Combining (1.11), (1.12) and (1.13) we get

$$\Gamma_i^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_n \end{smallmatrix}\right]\times 3} = \begin{cases} 1 & \text{if } i = 0, \\ (2^n - 1)\cdot 3 & \text{if } i = 1, \\ 7\cdot 2^{2n} - 9\cdot 2^n + 2 & \text{if } i = 2, \\ 2^{\sum_{j=1}^{n} s_j + 2n} - 7\cdot 2^{2n} + 6\cdot 2^n & \text{if } i = 3 \end{cases} \tag{1.14}$$

We get from (1.14), (1.9) and (1.4) whenever $s_j \geqslant 2$ for $1 \leqslant j \leqslant n$

$$\Gamma_i^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_n \end{smallmatrix}\right]\times 3} = \begin{cases} 1 & \text{if } i = 0, \\ \Gamma_1^{\left[\begin{smallmatrix} 1 \\ \vdots \\ 1 \end{smallmatrix}\right]\times 2} = (2^n - 1)\cdot 3 & \text{if } i = 1, \\ \Gamma_2^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times 3} = 7\cdot 2^{2n} - 9\cdot 2^n + 2 & \text{if } i = 2, \\ 2^{\sum_{j=1}^{n} s_j + 2n} - 7\cdot 2^{2n} + 6\cdot 2^n & \text{if } i = 3 \end{cases} \tag{1.15}$$

Example. We obtain from (1.9) with n=4:

$$\Gamma_i^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times 3} = \begin{cases} 1 & \text{if } i = 0, \\ 45 & \text{if } i = 1, \\ 1650 & \text{if } i = 2, \\ 63840 & \text{if } i = 3 \end{cases}$$

The number $\Gamma_i^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ (1) \end{smallmatrix}\right]\times 3}$ of rank i matrices of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 & \alpha_9 \\ \alpha_6 & \alpha_7 & \alpha_8 & \alpha_{19} & \alpha_{10} & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} & \alpha_{17} & \alpha_{18} & \alpha_{19} \end{pmatrix} \overset{\text{rank}}{\sim} \left(\begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_4 \\ \hline \\ \beta_1 & \beta_2 & \beta_3 \\ \beta_2 & \beta_3 & \beta_4 \\ \hline \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_4 \\ \hline \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \\ \hline \\ \delta_{11} & \delta_{12} & \delta_{13} \end{array}\right)$$

is equal to

$2^i\cdot\Gamma_i^{\left[\begin{smallmatrix}2\\2\\2\\2\end{smallmatrix}\right]\times 3} + (2^3-2^{i-1})\cdot\Gamma_i^{\left[\begin{smallmatrix}2\\2\\2\end{smallmatrix}\right]\times 3}$ $for \quad 0\leq i\leq inf(3,9)$ see [2]

That is

$$\Gamma_i^{\left[\begin{smallmatrix}2\\2\\2\\2\\(1)\end{smallmatrix}\right]\times 3} = \begin{cases}1 & \text{if } i=0,\\ 97 & \text{if } i=1,\\ 6870 & \text{if } i=2,\\ 5177320 & \text{if } i=3\end{cases}$$

1.3. **Computation of the number** $\Gamma_i^{\left[\begin{smallmatrix}s_1\\ \vdots\\ s_n\end{smallmatrix}\right]\times 4}$ **of n-times persymmetric** $(\sum_{j=1}^{n}s_j)\times 4$ **rank i matrices whenever** $s_j\geqslant 3$ **for** $1\leqslant j\leqslant n$**.** Denote by $D^{\left[\begin{smallmatrix}s_1\\ \vdots\\ s_n\end{smallmatrix}\right]\times 4}(t_1,t_2,\ldots,t_n)$

the following $(\sum_{j=1}^{n}s_j)\times 4$ n-times persymmetric matrix over the finite field $\mathbb{F}_2$

$$\left(\begin{array}{cccc}
\alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)}\\
\alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} & \alpha_5^{(1)}\\
\vdots & \vdots & \vdots & \\
\alpha_{s_1}^{(1)} & \alpha_{s_1+1}^{(1)} & \alpha_{s_1+2}^{(1)} & \alpha_{s_1+3}^{(1)}\\
\hline
\alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)}\\
\alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} & \alpha_5^{(2)}\\
\vdots & \vdots & \vdots & \\
\alpha_{s_2}^{(2)} & \alpha_{s_2+1}^{(2)} & \alpha_{s_2+2}^{(2)} & \alpha_{s_2+3}^{(2)}\\
\hline
\alpha_1^{(3)} & \alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)}\\
\alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} & \alpha_5^{(3)}\\
\vdots & \vdots & \vdots & \\
\alpha_{s_3}^{(3)} & \alpha_{s_3+1}^{(3)} & \alpha_{s_3+2}^{(3)} & \alpha_{s_3+3}^{(3)}\\
\hline
\vdots & \vdots & \vdots & \\
\hline
\alpha_1^{(n)} & \alpha_2^{(n)} & \alpha_3^{(n)} & \alpha_4^{(n)}\\
\alpha_2^{(n)} & \alpha_3^{(n)} & \alpha_4^{(n)} & \alpha_5^{(n)}\\
\vdots & \vdots & \vdots & \\
\alpha_{s_n}^{(n)} & \alpha_{s_n+1}^{(n)} & \alpha_{s_n+2}^{(n)} & \alpha_{s_n+3}^{(n)}
\end{array}\right)$$

Let $f(t_1,t_2,\ldots,t_n)$ be the exponential sum in $\mathbb{P}^n$ defined by
$(t_1,t_2,\ldots,t_n)\in\mathbb{P}^n \longrightarrow$
$\sum_{degY\leq 3}\sum_{degU_1\leq s_1-1} E(t_1YU_1)\sum_{degU_2\leq s_2-1} E(t_2YU_2)\ldots\sum_{degU_n\leq s_n-1} E(t_nYU_n).$

Then

$$f(t_1,t_2,\ldots,t_n) = 2^{\sum_{j=1}^{n}s_j+4-rank\left[D^{\left[\begin{smallmatrix}s_1\\ \vdots\\ s_n\end{smallmatrix}\right]\times 4}(t_1,t_2,\ldots,t_n)\right]}$$

Hence the number denoted by R_q of solutions

$$(Y_1,U_1^{(1)},U_2^{(1)},\ldots,U_n^{(1)},Y_2,U_1^{(2)},U_2^{(2)},\ldots,U_n^{(2)},\ldots Y_q,U_1^{(q)},U_2^{(q)},\ldots,U_n^{(q)})$$

of the polynomial equations

$$\begin{cases} Y_1U_1^{(1)}+Y_2U_1^{(2)}+\ldots+Y_qU_1^{(q)}=0\\ Y_1U_2^{(1)}+Y_2U_2^{(2)}+\ldots+Y_qU_2^{(q)}=0\\ \vdots\\ Y_1U_n^{(1)}+Y_2U_n^{(2)}+\ldots+Y_qU_n^{(q)}=0\end{cases}$$

satisfying the degree conditions

$$degY_i\leq 3,\quad degU_j^{(i)}\leq s_j-1,\quad for\quad 1\leq j\leq n\quad 1\leq i\leq q$$

is equal to the following integral over the unit interval in $\mathbb{K}^n$

$$\int_{\mathbb{P}^n} f^q(t_1,t_2,\ldots,t_n)dt_1dt_2\ldots dt_n.$$

Observing that $f(t_1,t_2,\ldots,t_n)$ is constant on cosets of $\prod_{j=1}^{n}\mathbb{P}_{s_j+3}$ in $\mathbb{P}^n$ the above integral is equal to

$$2^{q(\sum_{j=1}^{n}s_j+4)-\sum_{j=1}^{n}s_j-3\cdot n}\sum_{i=0}^{4}\Gamma_i^{\left[\begin{smallmatrix}s_1\\ \vdots\\ s_n\end{smallmatrix}\right]\times 4}2^{-iq}=R_q \tag{1.16}$$

From (1.16) we obtain for q = 1

$$2^{4-3n}\sum_{i=0}^{4}\Gamma_i^{\left[\begin{smallmatrix}s_1\\ \vdots\\ s_n\end{smallmatrix}\right]\times 4}2^{-i}=2^{\sum_{j=1}^{n}s_j}+2^4-1 \tag{1.17}$$

We have obviously

$$\sum_{i=0}^{4} \Gamma_i^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_1 \end{smallmatrix}\right]\times 4} = 2^{\sum_{j=1}^{n} s_j + 3n} \tag{1.18}$$

From the fact that the number of rank one persymmetric matrices over $\mathbb{F}_2$ is equal to three we obtain as above using combinatorial methods :

$$\Gamma_1^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_n \end{smallmatrix}\right]\times 4} = (2^n - 1)\cdot 3 \tag{1.19}$$

We assume now:

$$\Gamma_i^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_n \end{smallmatrix}\right]\times 4} = \begin{cases} 1 & \text{if } i = 0, \\ \Gamma_1^{\left[\begin{smallmatrix} 1 \\ \vdots \\ 1 \end{smallmatrix}\right]\times 2} = (2^n - 1)\cdot 3 & \text{if } i = 1, \\ \Gamma_2^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times 3} = 7\cdot 2^{2n} - 9\cdot 2^n + 2 & \text{if } i = 2, \end{cases} \tag{1.20}$$

To justify our assumption (1.20) for i=2 we remark that:

- The number of rank two persymmetric matrices over $\mathbb{F}_2$ is equal to $7\cdot 2^{2n} - 9\cdot 2^n + 2 = 7\cdot 2^2 - 9\cdot 2^1 + 2 = 12$ for n=1 [see (1), (2)]
- The number of rank two double persymmetric matrices over $\mathbb{F}_2$ is equal to $7\cdot 2^{2n} - 9\cdot 2^n + 2 = 7\cdot 2^4 - 9\cdot 2^2 + 2 = 78$ for n=2 [see (3)]
- The number of rank two triple persymmetric matrices over $\mathbb{F}_2$ is equal to $7\cdot 2^{2n} - 9\cdot 2^n + 2 = 7\cdot 2^6 - 9\cdot 2^3 + 2 = 378$ for n=3 [see (4)]

Combining (1.18), (1.19) and (1.20) we state that the number $\Gamma_i^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_n \end{smallmatrix}\right]\times 4}$ of n-times persymmetric $(\sum_{j=1}^{n} s_j)\times 4$ rank i matrices is equal to :

$$\begin{cases} 1 & \text{if } i = 0, \\ (2^n - 1)\cdot 3 & \text{if } i = 1, \\ 7\cdot 2^{2n} - 9\cdot 2^n + 2 & \text{if } i = 2, \\ 15\cdot 2^{3n} - 21\cdot 2^{2n} + 3\cdot 2^{n+1} & \text{if } i = 3, \\ 2^{\sum_{j=1}^{n} s_j + 3n} - 15\cdot 2^{3n} + 7\cdot 2^{2n+1} & \text{if } i = 4 \end{cases} \tag{1.21}$$

Example. We have for $n=3,\ s_1=s_2=s_3=3$.

$$\Gamma_i^{\left[\begin{smallmatrix}3\\3\\3\end{smallmatrix}\right]\times 4} = \begin{cases}1 & \text{if } i=0,\\ 21 & \text{if } i=1,\\ 378 & \text{if } i=2,\\ 6384 & \text{if } i=3,\\ 255360 & \text{if } i=4\end{cases}$$

See (4)

Example. We have for $n=4,\ s_1=s_2=s_3=4$.

$$\Gamma_i^{\left[\begin{smallmatrix}4\\4\\4\\4\end{smallmatrix}\right]\times 4} = \begin{cases}1 & \text{if } i=0,\\ 45 & \text{if } i=1,\\ 1650 & \text{if } i=2,\\ 56160 & \text{if } i=3,\\ 268377600 & \text{if } i=4\end{cases}$$

Hence the number R_4 of solutions

$(Y_1,U_1^{(1)},U_2^{(1)},U_3^{(1)},U_4^{(1)},Y_2,U_1^{(2)},U_2^{(2)},U_3^{(2)},U_4^{(2)},Y_3,U_1^{(3)},U_2^{(3)},U_3^{(3)},U_4^{(3)},Y_4,U_1^{(4)},U_2^{(4)},U_3^{(4)},U_4^{(4)}) \in (\mathbb{F}_2[T])^{20}$

of the polynomial equations

$$\begin{cases} Y_1U_1^{(1)}+Y_2U_1^{(2)}+Y_3U_1^{(3)}+Y_4U_1^{(4)}=0\\ Y_1U_2^{(1)}+Y_2U_2^{(2)}+Y_3U_2^{(3)}+Y_4U_2^{(4)}=0\\ Y_1U_3^{(1)}+Y_2U_3^{(2)}+Y_3U_3^{(3)}+Y_4U_3^{(4)}=0\\ Y_1U_4^{(1)}+Y_2U_4^{(2)}+Y_3U_4^{(3)}+Y_4U_4^{(4)}=0\end{cases}$$

satisfying the degree conditions

$$degY_i \leq 3, \quad degU_j^{(i)} \leq 3, \quad for \quad 1\leq j\leq 4, \quad 1\leq i\leq 4.$$

is equal to

$$2^{52}\cdot\sum_{i=0}^{4}\Gamma_i^{\left[\begin{smallmatrix}4\\4\\4\\4\end{smallmatrix}\right]\times 4}2^{-4i} = 2^{45}\cdot 527243$$

2. Computation of the number $\Gamma_{i}^{\left[\begin{smallmatrix} s_j \\ \vdots \\ s_j \end{smallmatrix}\right] \times k}$ of n-times persymmetric $(\sum_{j=1}^{n} s_j) \times k$ rank i matrices where $s_j \geqslant k-1$ for $1 \leqslant j \leqslant n$

Conjecture : Let $s_j \geqslant k-1$ for $1 \leqslant j \leqslant n$.
Set $(t_1, t_2, \ldots, t_n)$
$= \left(\sum_{i \geq 1} \alpha_i^{(1)} T^{-i}, \sum_{i \geq 1} \alpha_i^{(2)} T^{-i}, \sum_{i \geq 1} \alpha_i^{(3)} T^{-i}, \ldots, \sum_{i \geq 1} \alpha_i^{(n)} T^{-i} \right) \in \mathbb{P}^n$.
We state that the number $\Gamma_{i}^{\left[\begin{smallmatrix} s_j \\ \vdots \\ s_j \end{smallmatrix}\right] \times k}$ of rank i n- times persymmetric matrices over the finite field $\mathbb{F}_2$ of the below form denoted by $D^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_n \end{smallmatrix}\right] \times k}(t_1, t_2, \ldots, t_n)$

$$\left(\begin{array}{ccccc}
\alpha_1^{(1)} & \alpha_2^{(1)} & \ldots & \alpha_{k-1}^{(1)} & \alpha_k^{(1)} \\
\alpha_2^{(1)} & \alpha_3^{(1)} & \ldots & \alpha_k^{(1)} & \alpha_{k+1}^{(1)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{s_1}^{(1)} & \alpha_{s_1+1}^{(1)} & \ldots & \alpha_{s_1+k-2}^{(1)} & \alpha_{s_1+k-1}^{(1)} \\
\hline
\alpha_1^{(2)} & \alpha_2^{(2)} & \ldots & \alpha_{k-1}^{(2)} & \alpha_k^{(2)} \\
\alpha_2^{(2)} & \alpha_3^{(2)} & \ldots & \alpha_k^{(2)} & \alpha_{k+1}^{(2)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{s_2}^{(2)} & \alpha_{s_2+1}^{(2)} & \ldots & \alpha_{s_2+k-2}^{(2)} & \alpha_{s_2+k-1}^{(2)} \\
\hline
\alpha_1^{(3)} & \alpha_2^{(3)} & \ldots & \alpha_{k-1}^{(3)} & \alpha_k^{(3)} \\
\alpha_2^{(3)} & \alpha_3^{(3)} & \ldots & \alpha_k^{(3)} & \alpha_{k+1}^{(3)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{s_3}^{(3)} & \alpha_{s_3+1}^{(3)} & \ldots & \alpha_{s_3+k-2}^{(3)} & \alpha_{s_3+k-1}^{(3)} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
\alpha_1^{(n)} & \alpha_2^{(n)} & \ldots & \alpha_{k-1}^{(n)} & \alpha_k^{(n)} \\
\alpha_2^{(n)} & \alpha_3^{(n)} & \ldots & \alpha_k^{(n)} & \alpha_{k+1}^{(n)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{s_n}^{(n)} & \alpha_{s_n+1}^{(n)} & \ldots & \alpha_{s_n+k-2}^{(n)} & \alpha_{s_n+k-1}^{(n)}
\end{array} \right)$$

is equal to

(2.1)

$$\begin{cases} 1 & \text{if } i=0,\\ \Gamma_{i}^{\left[\begin{smallmatrix} i\\ \vdots \\ i\end{smallmatrix}\right]\times (i+1)} = (2^{i+1}-1)\cdot 2^{in}-3\cdot(2^{i}-1)\cdot 2^{(i-1)n}+(2^{i-1}-1)\cdot 2^{(i-2)n+1} & \\ = (2^{n}-1)\cdot(2^{n+1}-1)\cdot 2^{i(n+1)-2n}-(2^{n}-1)\cdot(2^{n-1}-1)\cdot 2^{in-2n+1} & \text{if } 1\leqslant i\leqslant k-1\\ 2^{\sum_{j=1}^{n}s_{j}+(k-1)n}-(2^{k}-1)\cdot 2^{(k-1)n}+(2^{k-1}-1)\cdot 2^{(k-2)n+1} & \text{if } i=k \end{cases}$$

where $\Gamma_{i}^{\left[\begin{smallmatrix} i\\ \vdots \\ i\end{smallmatrix}\right]\times (i+1)}$ denote the number of rank i n-times persymmetric matrices over $\mathbb{F}_2$ of the below form :

$$\left(\begin{array}{ccccc}
\alpha_1^{(1)} & \alpha_2^{(1)} & \ldots & \alpha_i^{(1)} & \alpha_{i+1}^{(1)}\\
\alpha_2^{(1)} & \alpha_3^{(1)} & \ldots & \alpha_{i+1}^{(1)} & \alpha_{i+2}^{(1)}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
\alpha_i^{(1)} & \alpha_{i+1}^{(1)} & \ldots & \alpha_{2i-2}^{(1)} & \alpha_{2i-1}^{(1)}\\
\hline
\alpha_1^{(2)} & \alpha_2^{(2)} & \ldots & \alpha_i^{(2)} & \alpha_{i+1}^{(2)}\\
\alpha_2^{(2)} & \alpha_3^{(2)} & \ldots & \alpha_{i+1}^{(2)} & \alpha_{i+2}^{(2)}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
\alpha_i^{(2)} & \alpha_{i+1}^{(2)} & \ldots & \alpha_{2i-2}^{(2)} & \alpha_{2i-1}^{(2)}\\
\hline
\alpha_1^{(3)} & \alpha_2^{(3)} & \ldots & \alpha_i^{(3)} & \alpha_{i+1}^{(3)}\\
\alpha_2^{(3)} & \alpha_3^{(3)} & \ldots & \alpha_{i+1}^{(3)} & \alpha_{i+2}^{(3)}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
\alpha_i^{(3)} & \alpha_{i+1}^{(3)} & \ldots & \alpha_{2i-2}^{(3)} & \alpha_{2i-1}^{(3)}\\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots\\
\hline
\alpha_1^{(n)} & \alpha_2^{(n)} & \ldots & \alpha_i^{(n)} & \alpha_{i+1}^{(n)}\\
\alpha_2^{(n)} & \alpha_3^{(n)} & \ldots & \alpha_{i+1}^{(n)} & \alpha_{i+2}^{(n)}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
\alpha_i^{(n)} & \alpha_{i+1}^{(n)} & \ldots & \alpha_{2i-2}^{(n)} & \alpha_{2i-1}^{(n)}
\end{array}\right)$$

Application :

The number denoted by R_q of solutions

$(Y_1,U_1^{(1)},U_2^{(1)},\ldots,U_n^{(1)},Y_2,U_1^{(2)},U_2^{(2)},\ldots,U_n^{(2)},\ldots Y_q,U_1^{(q)},U_2^{(q)},\ldots,U_n^{(q)})\in (\mathbb{F}_2[T])^{(n+1)q}$

of the polynomial equations

$$\begin{cases} Y_1U_1^{(1)} + Y_2U_1^{(2)} + \ldots + Y_qU_1^{(q)} = 0 \\ Y_1U_2^{(1)} + Y_2U_2^{(2)} + \ldots + Y_qU_2^{(q)} = 0 \\ \vdots \\ Y_1U_n^{(1)} + Y_2U_n^{(2)} + \ldots + Y_qU_n^{(q)} = 0 \end{cases}$$

satisfying the degree conditions

$$degY_i \leq k-1, \quad degU_j^{(i)} \leq s_j-1 \text{ where } s_j \geqslant k-1 \quad for \quad 1 \leq j \leq n, \quad 1 \leq i \leq q$$

is equal to

$$2^{q(\sum_{j=1}^{n} s_j+k)-\sum_{j=1}^{n} s_j-(k-1)\cdot n}\sum_{i=0}^{k}\Gamma_i^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_n \end{smallmatrix}\right]\times k} 2^{-iq} = R_q \tag{2.2}$$

Conditional proof :

Inspired by the results in subsection 1.3 we proceed as follows :

We assume that $\Gamma_i^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_n \end{smallmatrix}\right]\times k}$ is equal to

(2.3)

$$\begin{cases} 1 & \text{if } i=0, \\ \Gamma_1^{\left[\begin{smallmatrix} 1 \\ \vdots \\ 1 \end{smallmatrix}\right]\times 2} = (2^n-1)\cdot 3 & \text{if } i=1, \\ \Gamma_2^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times 3} = 7\cdot 2^{2n} - 9\cdot 2^n + 2 & \text{if } i=2, \\ \Gamma_i^{\left[\begin{smallmatrix} i \\ \vdots \\ i \end{smallmatrix}\right]\times (i+1)} = (2^{i+1}-1)\cdot 2^{in} - 3\cdot(2^i-1)\cdot 2^{(i-1)n} + (2^{i-1}-1)\cdot 2^{(i-2)n+1} & \\ & \text{if } 1 \leqslant i \leqslant k-2 \end{cases}$$

To justify our assumption (2.3) we remark that our supposition is valid for n equal to one, two and three.

(1) **The case n =1**

The number of rank i persymmetric matrices over $\mathbb{F}_2$ of the form

$$\begin{pmatrix} \alpha_1^{(1)} & \alpha_2^{(1)} & \ldots & \alpha_{k-1}^{(1)} & \alpha_k^{(1)} \\ \alpha_2^{(1)} & \alpha_3^{(1)} & \ldots & \alpha_k^{(1)} & \alpha_{k+1}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{s_1}^{(1)} & \alpha_{s_1+1}^{(1)} & \ldots & \alpha_{s_1+k-2}^{(1)} & \alpha_{s_1+k-1}^{(1)} \end{pmatrix}$$

is equal to

$(2^{i+1}-1)\cdot 2^i - 3\cdot(2^i-1)\cdot 2^{i-1} + (2^{i-1}-1)\cdot 2^{i-1} = 3\cdot 2^{2i-2}$ [see (1), (2)]

(2) **The case n =2**
The number of rank i double persymmetric matrices over $\mathbb{F}_2$ of the form

$$\left(\begin{array}{ccccc}
\alpha_1^{(1)} & \alpha_2^{(1)} & \ldots & \alpha_{k-1}^{(1)} & \alpha_k^{(1)} \\
\alpha_2^{(1)} & \alpha_3^{(1)} & \ldots & \alpha_k^{(1)} & \alpha_{k+1}^{(1)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{s_1}^{(1)} & \alpha_{s_1+1}^{(1)} & \ldots & \alpha_{s_1+k-2}^{(1)} & \alpha_{s_1+k-1}^{(1)} \\
\hline
\alpha_1^{(2)} & \alpha_2^{(2)} & \ldots & \alpha_{k-1}^{(2)} & \alpha_k^{(2)} \\
\alpha_2^{(2)} & \alpha_3^{(2)} & \ldots & \alpha_k^{(2)} & \alpha_{k+1}^{(2)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{s_2}^{(2)} & \alpha_{s_2+1}^{(2)} & \ldots & \alpha_{s_2+k-2}^{(2)} & \alpha_{s_2+k-1}^{(2)}
\end{array}\right)$$

is equal to

$(2^{i+1}-1)\cdot 2^{2i} - 3\cdot(2^{i}-1)\cdot 2^{2(i-1)} + (2^{i-1}-1)\cdot 2^{2(i-2)+1}$
$= 21\cdot 2^{3i-4} - 3\cdot 2^{2i-3}$ [see(3)]

(3) **The case n =3**
The number of rank i triple persymmetric matrices over $\mathbb{F}_2$ of the form

$$\left(\begin{array}{ccccc}
\alpha_1^{(1)} & \alpha_2^{(1)} & \ldots & \alpha_{k-1}^{(1)} & \alpha_k^{(1)} \\
\alpha_2^{(1)} & \alpha_3^{(1)} & \ldots & \alpha_k^{(1)} & \alpha_{k+1}^{(1)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{s_1}^{(1)} & \alpha_{s_1+1}^{(1)} & \ldots & \alpha_{s_1+k-2}^{(1)} & \alpha_{s_1+k-1}^{(1)} \\
\hline
\alpha_1^{(2)} & \alpha_2^{(2)} & \ldots & \alpha_{k-1}^{(2)} & \alpha_k^{(2)} \\
\alpha_2^{(2)} & \alpha_3^{(2)} & \ldots & \alpha_k^{(2)} & \alpha_{k+1}^{(2)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{s_2}^{(2)} & \alpha_{s_2+1}^{(2)} & \ldots & \alpha_{s_2+k-2}^{(2)} & \alpha_{s_2+k-1}^{(2)} \\
\hline
\alpha_1^{(3)} & \alpha_2^{(3)} & \ldots & \alpha_{k-1}^{(3)} & \alpha_k^{(3)} \\
\alpha_2^{(3)} & \alpha_3^{(3)} & \ldots & \alpha_k^{(3)} & \alpha_{k+1}^{(3)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{s_3}^{(3)} & \alpha_{s_3+1}^{(3)} & \ldots & \alpha_{s_3+k-2}^{(3)} & \alpha_{s_3+k-1}^{(3)}
\end{array}\right)$$

is equal to
$(2^{i+1}-1)\cdot 2^{3i} - 3\cdot(2^{i}-1)\cdot 2^{3(i-1)} + (2^{i-1}-1)\cdot 2^{3(i-2)+1}$
$= 105\cdot 2^{4i-6} - 21\cdot 2^{3i-5}$[see (4),(5)]

Let $f(t_1,t_2,\ldots,t_n)$ be the exponential sum in $\mathbb{P}^n$ defined by
$(t_1,t_2,\ldots,t_n)\in\mathbb{P}^n \longrightarrow$

$$\sum_{degY\leq k-1}\sum_{degU_1\leq s_1-1}E(t_1YU_1)\sum_{degU_2\leq s_2-1}E(t_2YU_2)\ldots\sum_{degU_n\leq s_n-1}E(t_nYU_n).$$

Then

$$f(t_1,t_2,\dots,t_n) = 2^{\sum_{j=1}^n s_j + k - rank\left[D^{\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}\times k}(t_1,t_2,\dots,t_n)\right]}$$

Hence the number R_q is equal to the following integral over the unit interval in $\mathbb{K}^n$

$$\int_{\mathbb{P}^n} f^q(t_1,t_2,\dots,t_n)dt_1 dt_2 \dots dt_n.$$

Observing that $f(t_1,t_2,\dots,t_n)$ is constant on cosets of $\prod_{j=1}^n \mathbb{P}_{s_j+k-1}$ in $\mathbb{P}^n$ the above integral is equal to

$$2^{q(\sum_{j=1}^n s_j + k) - \sum_{j=1}^n s_j - (k-1)\cdot n} \sum_{i=0}^{k} \Gamma_i^{\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}\times k} 2^{-iq} = R_q \tag{2.4}$$

From (2.4) we obtain for q = 1

$$2^{k-(k-1)n} \sum_{i=0}^{k} \Gamma_i^{\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}\times k} 2^{-i} = 2^{\sum_{j=1}^n s_j} + 2^k - 1 \tag{2.5}$$

We have obviously

$$\sum_{i=0}^{k} \Gamma_i^{\begin{bmatrix} s_1 \\ \vdots \\ s_1 \end{bmatrix}\times k} = 2^{\sum_{j=1}^n s_j + (k-1)n} \tag{2.6}$$

From (2.6) and our assumption (2.3) we get :

$$\Gamma_{k-1}^{\begin{bmatrix} s_1 \\ \vdots \\ s_1 \end{bmatrix}\times k} + \Gamma_{k}^{\begin{bmatrix} s_1 \\ \vdots \\ s_1 \end{bmatrix}\times k} = 2^{\sum_{j=1}^n s_j + (k-1)n} - \sum_{i=0}^{k-2} \Gamma_i^{\begin{bmatrix} s_1 \\ \vdots \\ s_1 \end{bmatrix}\times k} \tag{2.7}$$

$$= 2^{\sum_{j=1}^n s_j + (k-1)n} - [2^{(k-2)n}\cdot(2^{k-1}-1) + 2^{(k-3)n}\cdot(2-2^{k-1})]$$

From (2.5) and our assumption (2.3) we obtain :

(2.8)

$$2\cdot\Gamma_{k-1}^{\begin{bmatrix} s_1 \\ \vdots \\ s_1 \end{bmatrix}\times k} + \Gamma_{k}^{\begin{bmatrix} s_1 \\ \vdots \\ s_1 \end{bmatrix}\times k} = 2^{\sum_{j=1}^n s_j + (k-1)n} + 2^{(k-1)n}\cdot[2^k-1] - \sum_{i=0}^{k-2} 2^{k-i}\Gamma_i^{\begin{bmatrix} s_1 \\ \vdots \\ s_1 \end{bmatrix}\times k}$$

$$= 2^{\sum_{j=1}^{n} s_j+(k-1)n} + 2^{(k-1)n} \cdot [2^k - 1] + 2^{nk-3n} \cdot [2^k - 2^2] + 2^{nk-2n} \cdot [2^2 - 2^{k+1}]$$

We deduce from (2.7) and (2.8) :

$$(2.9)\quad \Gamma_{k-1}^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_1 \end{smallmatrix}\right] \times k} = 2^{(k-1)n} \cdot [2^k - 1] + 3 \cdot 2^{nk-2n} \cdot [1 - 2^{k-1}] + 2^{nk-3n+1} \cdot [2^{k-2} - 1]$$

$$(2.10)\quad \Gamma_{k}^{\left[\begin{smallmatrix} s_1 \\ \vdots \\ s_1 \end{smallmatrix}\right] \times k} = 2^{\sum_{j=1}^{n} s_j+(k-1)n} - (2^k - 1) \cdot 2^{(k-1)n} + (2^{k-1} - 1) \cdot 2^{(k-2)n+1}.$$

References

[1] Daykin David E, Distribution of Bordered Persymmetric Matrices in a finite field J. reine angew. Math, **203**(1960) ,47-54
[2] Cherly, Jorgen.
Exponential sums and rank of persymmetric matrices over $\mathbf{F}_2$
arXiv : 0711.1306, 46 pp
[3] Cherly, Jorgen.
Exponential sums and rank of double persymmetric matrices over $\mathbf{F}_2$
arXiv : 0711.1937, 160 pp
[4] Cherly, Jorgen.
Exponential sums and rank of triple persymmetric matrices over $\mathbf{F}_2$
arXiv : 0803.1398, 233 pp
[5] Cherly, Jorgen.
Results about persymmetric matrices over $\mathbf{F}_2$ and related exponentials sums
arXiv : 0803.2412v2, 32 pp
[6] Cherly, Jorgen.
Polynomial equations and rank of matrices over $\mathbf{F}_2$ related to persymmetric matrices
arXiv : 0909.0438v1, 33 pp

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